

# Engineering Mathematics-II (BT-202) Module-4 Notes

## Functions of Complex Variable

### MODULE-4: Functions of Complex Variable

#### Topics Covered

- Functions of Complex Variables
- Analytic Functions
- Harmonic Conjugate
- Cauchy-Riemann Equations
- Line Integral
- Cauchy-Goursat Theorem
- Cauchy Integral Formula
- Singular Points
- Poles and Residues
- Residue Theorem
- Application of Residue Theorem in Real Integrals

#### 1. COMPLEX VARIABLE

A complex variable is represented as:

$$z = x + iy$$

Where:

$x$  = Real part

$y$  = Imaginary part

$i = \sqrt{-1}$

#### 2. FUNCTION OF COMPLEX VARIABLE

A function of complex variable is represented as:

$$f(z) = u(x,y) + iv(x,y)$$

Where:

$u(x,y)$  = Real function

$v(x,y)$  = Imaginary function

#### 3. ANALYTIC FUNCTION

A function is analytic if derivative exists at every point in a region.

Condition for analyticity:

Function must satisfy Cauchy-Riemann equations.

#### 4. CAUCHY-RIEMANN EQUATIONS

If:

$$f(z) = u + iv$$

Then C-R equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If these equations are satisfied and derivatives are continuous, then function is analytic.

#### 5. HARMONIC FUNCTION

A function satisfying Laplace equation is harmonic.

Laplace Equation:

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0$$

## 6. HARMONIC CONJUGATE

If  $u$  and  $v$  satisfy C-R equations then  $v$  is harmonic conjugate of  $u$ .

**Steps to Find Harmonic Conjugate:**

1. Apply C-R equations
2. Integrate partially
3. Find unknown function

## 7. LINE INTEGRAL

Integral of complex function along a curve is called line integral.

Formula:

$$\int f(z) dz$$

Where:

$$dz = dx + idy$$

## 8. CAUCHY-GOURSAT THEOREM

If function is analytic inside and on a closed contour  $C$ , then:

$$\oint_C f(z) dz = 0$$

**Applications:**

- Evaluation of integrals
- Complex analysis problems

## 9. CAUCHY INTEGRAL FORMULA

If  $f(z)$  is analytic inside contour  $C$  and  $a$  is inside  $C$ , then:

$$f(a) = \frac{1}{2\pi i} \oint_C [f(z)/(z-a)] dz$$

**Applications:**

- Finding derivatives
- Evaluating contour integrals

## 10. SINGULAR POINTS

A point where function is not analytic is called singular point.

**Types:**

- Isolated singularity
- Non-isolated singularity

## 11. POLES

A singularity where function becomes infinite is called pole.

**Types of Poles:**

- Simple pole
- Pole of order  $n$

## 12. RESIDUE

Coefficient of  $1/(z-a)$  in Laurent series expansion is residue.

### Formula for Simple Pole:

$$\text{Residue} = \lim_{z \rightarrow a} (z-a)f(z)$$

## 13. RESIDUE THEOREM

If  $f(z)$  has poles inside contour  $C$ , then:

$$\oint_C f(z) dz = 2\pi i \times \sum \text{Residues}$$

### Applications:

- Evaluation of contour integrals
- Evaluation of real integrals

## 14. EVALUATION OF REAL INTEGRALS USING RESIDUE THEOREM

Unit circle method is used for integrals involving trigonometric functions.

Substitution:

$$z = e^{i\theta}$$

Then:

$$\cos\theta = (z + 1/z)/2$$

$$\sin\theta = (z - 1/z)/(2i)$$

## 15. IMPORTANT RESULTS

1. If function is analytic then it satisfies C-R equations.
2. Harmonic functions satisfy Laplace equation.
3. Residue theorem simplifies contour integrals.

### MOST IMPORTANT 14 MARK QUESTIONS

1. Verify analyticity using Cauchy-Riemann equations.
2. Find harmonic conjugate of given function.
3. Evaluate complex line integrals.
4. State and apply Cauchy-Goursat theorem.
5. State and apply Cauchy Integral Formula.
6. Find singular points and poles of complex functions.
7. Find residues at poles.
8. Apply residue theorem to evaluate contour integrals.
9. Evaluate real integrals using residue theorem and unit circle method.
10. Discuss poles and residues with examples.

### IMPORTANT 7 MARK QUESTIONS

1. Define analytic function.
2. State Cauchy-Riemann equations.
3. Define harmonic function.
4. Define harmonic conjugate.
5. Define line integral.
6. State Cauchy-Goursat theorem.
7. State Cauchy Integral Formula.
8. Define singular point.
9. Define pole and residue.
10. State residue theorem.

### **EXAM TIPS**

- Practice C-R equation problems daily.
- Learn residue formulas carefully.
- Practice contour integration numericals.
- Revise poles and singularities regularly.
- Focus on unit circle substitution problems.